

Local Gauge Transformation for the Quark Propagator in an $SU(N)$ Gauge Theory

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In an $SU(N)$ gauge field theory, the n -point Green functions, namely, propagators and vertices, transform under the simultaneous local gauge variations of the gluon vector potential and the quark matter field in such a manner that the physical observables remain invariant. In this article, we derive this intrinsically non perturbative transformation law for the quark propagator within the system of covariant gauges. We carry out its explicit perturbative expansion till $\mathcal{O}(g_s^6)$ and, for some terms, till $\mathcal{O}(g_s^8)$. We study the implications of this transformation for the quark-anti-quark condensate, multiplicative renormalizability of the massless quark propagator, as well as its relation with the quark-gluon vertex at the one-loop order. Setting the color factors $C_F = 1$ and $C_A = 0$, Landau-Khalatnikov-Fradkin transformation for the abelian case of quantum electrodynamics is trivially recovered.

I. INTRODUCTION

In gauge field theories, Green functions transform in a specific manner under a variation of gauge. In quantum electrodynamics (QED), these transformations carry the name of Landau-Khalatnikov-Fradkin transformations (LKFT) due to their first derivation by these authors in the canonical formulation of QED, [1]. Later, these were also derived by Johnson and Zumino through functional methods, [2]. LKFT are non perturbative in nature just like Ward-Fradkin-Green-Takahashi identities (WFGTI), [3]. The former govern the behavior of a single Green function when variation of covariant gauge parameter is performed, whereas, the latter relate different Green functions under local gauge transformations.

WFGTI follow from the so called Becchi-Rouet-Stora-Tyutin (BRST) symmetry. One can further enlarge these transformations by modifying the Lagrangian without affecting the dynamics of the theory. Now also transforming the gauge parameter [4], one can arrive at extended Ward identities, known as Nielsen identities (NI). An advantage of the NI over the conventional Ward identities is that $\partial/\partial\xi$ becomes an integral part of the new relations involving Green functions. This fact was exploited in [6] to prove the gauge independence of the fermion pole mass related to the two-point Green functions.

Although LKFT and WFGTI are independent relations, one can show that if a given WFGTI holds in one gauge, then if the Green functions involved are transformed to another gauge under LKFT, the WFGTI in that other gauge will be satisfied automatically and need not be rechecked. In general, the rules governing LKFT are far from simple. The fact that they are written in coordinate space adds to their complexity. As a result,

these transformations have played a less significant and practical role in the study of gauge theories as compared to WFGTI.

In perturbation theory, both the WFGTI and the LKFT are satisfied at every order of approximation. This perturbative approach is sufficient to study QED even at the highest energies so far achieved in experiments. The only gauge theory whose non perturbative understanding is relevant to fundamental interactions within the standard model is quantum chromodynamics (QCD). In the infrared domain of QCD, emergent phenomena like dynamical chiral symmetry breaking (DCSB) and confinement are observed which cannot be accessed through the usual perturbative tools. Their gauge invariance needs to be established in any physically acceptable formulation of QCD. The gauge identities which relate Green functions in QCD are called Slavnov-Taylor identities (STI), [5]. These identities play a key role in the proof of gauge-invariant renormalizability of QCD.

However, generalized LKFT for QCD have not been derived. Making sure that gauge invariance remains intact in the non perturbative studies of QCD is a highly non-trivial matter. Therefore, it is important to have all relevant tools at hand, including the non-abelian version of LKFT. Algebraically, QCD is not too different from any other $SU(N)$ theory for our purpose. So we set out to derive this transformation for the quark propagator in an arbitrary $SU(N)$ gauge theory. Later, we can specialize to QCD by setting the color factors $C_F = 4/3$, $C_A = 3$ or even to QED for the choice of $C_F = 1$ and $C_A = 0$.

We have organized this article as follows. In Sect. II, we derive the local gauge transformation relation for the quark propagator in the coordinate space. We provide

its explicit perturbative expansion till $\mathcal{O}(g_s^6)$ and, in case of some terms, till $\mathcal{O}(g_s^8)$. Sect. III is devoted to the implications of these relations in perturbation theory in momentum space, discussing the relation of the generalized LKFT with multiplicative renormalizability of the massless quark propagator. Sect. IV details the role of the quark-gluon vertex in ensuring the multiplicative renormalizability of the quark propagator and hence its connection with the LKFT. In Sect. V, we present conclusions and some possible avenues of future work.

II. QUARK PROPAGATOR UNDER LOCAL $SU(N)$ GAUGE TRANSFORMATIONS

Local $SU(N)$ gauge transformation for the quark field reads as :

$$\psi_i(x) \rightarrow \psi'_i(x) = e^{i[g_s \varphi_a(x) T_a]} \psi_i(x), \quad (1)$$

where $\varphi_a(x)$ is an arbitrary free scalar field operator carrying color quantum number a ($a = 1, 2, \dots, N^2 - 1$), g_s is the strong coupling constant and $T_a = \lambda_a/2$, λ_a being the well-known Gell-Mann matrices for $N = 3$. Following the work of Landau and Khalatnikov [1] for QED, we shall investigate how the quark Green function (quark propagator in the coordinate space) would transform under this gauge transformation. The quark Green function is defined as :

$$iS_{ij}^F(x, x') \equiv iS_{ij}^F = \langle T\{\psi_i(x) \bar{\psi}_j(x')\} \rangle, \quad (2)$$

where the angle brackets stand for the vacuum expectation value. Under local $SU(N)$ local gauge transformation of Eq. (1), we have

$$iS_{ij}^F(x, x') \rightarrow \langle T\{e^{i[g_s \varphi_a(x) T_a]} \psi_i(x) \bar{\psi}_j(x') e^{-i[g_s \varphi_b(x') T_b]}\} \rangle. \quad (3)$$

Note that the color matrices act on the corresponding fermion fields. If we explicitly show the color indices, we can write :

$$\begin{aligned} (T_a \psi_i) (\bar{\psi}_j T_b) & [\Rightarrow ((T_a \psi_i) (\bar{\psi}_j T_b))_{\sigma\mu}] \\ &= (T_a)_{\sigma\lambda} (\psi_i)_\lambda (\bar{\psi}_j)_\nu (T_b)_{\nu\mu} = [(T_a)_{\sigma\lambda} (T_b)_{\nu\mu}] (\psi_i)_\lambda (\bar{\psi}_j)_\nu. \end{aligned}$$

The vacuum expectation value yields a Kronecker delta-function for the color indices of the fermion fields, i.e., $\delta_{\lambda\nu}$. Thus we are left with (for the color part)

$$[(T_a)_{\sigma\lambda} (T_b)_{\nu\mu}] \delta_{\lambda\nu} = (T_a)_{\sigma\nu} (T_b)_{\nu\mu} = [T_a T_b]_{\sigma\mu}.$$

The same will be true for any number of generators multiplying together. So we omit the explicit color indices on the generating matrices as well as on the quark fields and indicate only the Lorentz indices of the quark fields, namely i, j . Since there are two fermion and two scalar fields, the only surviving contractions get factorized as :

$$iS_{ij}^F(x, x') \rightarrow iS_{ij}^{0F}(x, x') \langle T\{e^{i[g_s \varphi_a(x) T_a]} e^{-i[g_s \varphi_b(x') T_b]}\} \rangle, \quad (4)$$

where $S_{ij}^{0F}(x, x')$ stands for the Green function for quarks for the particular case when the longitudinal part of the

gluon's Green function is zero (Landau gauge). Note that this expression for a non-abelian $SU(N)$ theory is inherently different from the corresponding abelian expression which has no non-commuting operators in the exponentials and hence can trivially be combined. In QCD, the non-commuting part of the algebra (gluon self-interactions) introduces the color factor C_A which enters only at $\mathcal{O}(g_s^4)$ in the quark propagator expansion. In the abelian approximation of QCD, i.e., $C_A = 0$, the final result resembles that of QED with C_F sitting as an overall multiplication factor:

$$iS_{ij}^F(x, x') = iS_{ij}^{0F}(x, x') e^{g_s^2 C_F [i\Delta_F(x-x') - i\Delta_F(0)]}. \quad (5)$$

On including the gluon self interactions, one can either use the Bakker-Cambell-Hausdorff formula for combining the non-commuting group generators or expand out the exponential, use the color algebra to simplify the results and look for patterns in the emerging series expansion to identify them with well-known exponential expressions.

The odd power terms in the scalar field are annihilated on taking the vacuum expectation value and hence always drop out. We employ the following notation for the Green functions associated with the scalar fields :

$$i\Delta_F(x - x') \delta_{ab} = \langle 0 | T\{\varphi_a(x) \varphi_b(x')\} | 0 \rangle. \quad (6)$$

We confirm the abelian-like result, Eq. (5) by expanding out Eq. (4) till $\mathcal{O}(g_s^8)$. It is done by employing the Wick theorem on the time ordered products and evaluating all the contractions involved in the series. The symmetry pattern emerging at every increasing order of perturbation theory is:

$$C_F^n \Rightarrow \frac{1}{n!} [i\Delta_F(x - x') - i\Delta_F(0)]^n. \quad (7)$$

This pattern requires a careful identification of the C_F factors in general because both C_F and C_A involve the number of colors N . Additionally, the factor $[i\Delta_F(x - x') - i\Delta_F(0)]$ is also crucial to guarantee the gauge invariance of the chiral quark condensate as we shall see shortly.

The quark gluon interaction as well as the gluon 3- and 4-point interactions introduce factors of $C_F - C_A/2$ (e.g., abelian type one-loop quark-gluon vertex) and C_A (e.g., non-abelian one-loop quark-gluon vertex, as well as gluon and ghost loops) in an intricate manner at increasing order of perturbation theory, resulting in non trivial coefficients of $i\Delta_F(x - x')$, $i\Delta_F(0)$ and their overlapping terms. We have calculated these additional terms to $\mathcal{O}(g_s^8)$. We present some defining steps of the calculation till $\mathcal{O}(g_s^4)$. As the details can be cumbersome, we shall only give the final expression till $\mathcal{O}(g_s^6)$. We do not give all the explicit expressions at $\mathcal{O}(g_s^8)$ because these terms do not add new features to the general form of the LKFT unless we mention it otherwise.

Let us calculate the term involving the exponentials :

$$\begin{aligned}
& \langle 0 | T \{ e^{i[g_s \varphi_a(x) T_a]} e^{-i[g_s \varphi_b(x') T_b]} \} | 0 \rangle \\
&= \langle 0 | T \left\{ \left[1 + i g_s \varphi_a(x) T_a + \frac{(i g_s)^2}{2!} (\varphi_a(x) T_a)^2 + \frac{(i g_s)^3}{3!} (\varphi_a(x) T_a)^3 + \frac{(i g_s)^4}{4!} (\varphi_a(x) T_a)^4 + \dots \right] \right. \\
&\quad \times \left. \left[1 - i g_s \varphi_b(x') T_b + \frac{(-i g_s)^2}{2!} (\varphi_b(x') T_b)^2 + \frac{(-i g_s)^3}{3!} (\varphi_b(x') T_b)^3 + \frac{(-i g_s)^4}{4!} (\varphi_b(x') T_b)^4 + \dots \right] \right\} | 0 \rangle.
\end{aligned}$$

We keep terms till $\mathcal{O}(g_s^4)$. Note that only terms of the type: $\varphi^0, \varphi^2, \varphi^4$ survive. The linear and cubic terms in the scalar field drop out when averaged in the vacuum state :

$$\begin{aligned}
& \langle 0 | T \{ e^{i[g_s \varphi_a(x) T_a]} e^{-i[g_s \varphi_b(x') T_b]} \} | 0 \rangle \\
&= \langle 0 | T \left\{ 1 + g_s^2 [\varphi_a(x) \varphi_b(x')] T_a T_b - \frac{g_s^2}{2} [\varphi_a(x) \varphi_b(x)] T_a T_b - \frac{g_s^2}{2} [\varphi_a(x') \varphi_b(x')] T_a T_b \right. \\
&\quad + \frac{g_s^4}{4!} [\varphi_a(x) \varphi_b(x) \varphi_c(x) \varphi_d(x)] T_a T_b T_c T_d + \frac{g_s^4}{4!} [\varphi_a(x') \varphi_b(x') \varphi_c(x') \varphi_d(x')] T_a T_b T_c T_d \\
&\quad - \frac{g_s^4}{3!} [\varphi_a(x) \varphi_b(x) \varphi_c(x) \varphi_d(x')] T_a T_b T_c T_d - \frac{g_s^4}{3!} [\varphi_a(x) \varphi_b(x') \varphi_c(x') \varphi_d(x')] T_a T_b T_c T_d \\
&\quad \left. + \frac{g_s^4}{4} [\varphi_a(x) \varphi_b(x) \varphi_c(x') \varphi_d(x')] T_a T_b T_c T_d + \mathcal{O}(g_s^6) \right\} | 0 \rangle. \tag{8}
\end{aligned}$$

Note that at $\mathcal{O}(g_s^4)$, quark propagator involves self interactions of gluons which bring in factors of C_A (the quark gluon vertex will contribute factors of $C_F - C_A/2$ for the abelian diagram and C_A for the non-abelian diagram including 3-point self gluon interactions). The ghost and gluon loop will also contribute a factor of C_A . These factors do not have a counterpart in QED as photons are not self-interacting. Therefore, we would like to detail the $\mathcal{O}(g_s^4)$ calculation. We proceed to do it as follows :

$$\begin{aligned}
& \langle 0 | T \{ \varphi_a(x) \varphi_b(x) \varphi_c(x) \varphi_d(x) \} | 0 \rangle T_a T_b T_c T_d \\
&= \left[\langle 0 | \overline{\varphi_a(x) \varphi_b(x)} \overline{\varphi_c(x) \varphi_d(x)} | 0 \rangle + \langle 0 | \overline{\varphi_a(x) \varphi_b(x) \varphi_c(x) \varphi_d(x)} | 0 \rangle + \langle 0 | \overline{\varphi_a(x) \varphi_b(x) \varphi_c(x) \varphi_d(x)} | 0 \rangle \right] T_a T_b T_c T_d \\
&= i \Delta_F(0) i \Delta_F(0) [\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}] T_a T_b T_c T_d = (i \Delta_F(0))^2 [T_a T_a T_c T_c + T_a T_b T_a T_b + T_a T_b T_b T_a].
\end{aligned}$$

It is the middle term in the square bracket of the last term which brings in the non-abelian nature of QCD as two identical T_i are not next to each other. Using the color identities collected in the appendix, we arrive at the following results:

$$\begin{aligned}
& \langle 0 | T \{ \varphi_a(x) \varphi_b(x) \varphi_c(x) \varphi_d(x) \} | 0 \rangle T_a T_b T_c T_d = (i \Delta_F(0))^2 \left[3C_F^2 - \frac{1}{2} C_A C_F \right] \\
& \langle 0 | T \{ \varphi_a(x') \varphi_b(x') \varphi_c(x') \varphi_d(x') \} | 0 \rangle T_a T_b T_c T_d = (i \Delta_F(0))^2 \left[3C_F^2 - \frac{1}{2} C_A C_F \right] \\
& \langle 0 | T \{ \varphi_a(x) \varphi_b(x) \varphi_c(x) \varphi_d(x') \} | 0 \rangle T_a T_b T_c T_d = (i \Delta_F(0)) (i \Delta_F(x - x')) \left[3C_F^2 - \frac{1}{2} C_A C_F \right] \\
& \langle 0 | T \{ \varphi_a(x) \varphi_b(x') \varphi_c(x') \varphi_d(x') \} | 0 \rangle T_a T_b T_c T_d = (i \Delta_F(0)) (i \Delta_F(x - x')) \left[3C_F^2 - \frac{1}{2} C_A C_F \right] \\
& \langle 0 | T \{ \varphi_a(x) \varphi_b(x) \varphi_c(x') \varphi_d(x') \} | 0 \rangle T_a T_b T_c T_d = (i \Delta_F(0))^2 C_F^2 + (i \Delta_F(x - x'))^2 \left[2C_F^2 - \frac{1}{2} C_A C_F \right]. \tag{9}
\end{aligned}$$

We can now simplify the $\mathcal{O}(g_s^4)$ terms in expression (8) and collect the coefficients of $(i \Delta_F(0))^2$, $(i \Delta_F(x - x'))^2$ and $(i \Delta_F(0)) (i \Delta_F(x - x'))$ to arrive at the final expression to $\mathcal{O}(g_s^4)$:

$$\begin{aligned}
i S_{ij}^F(x, x') &= i S_{ij}^{0F}(x, x') \left[e^{g_s^2 C_F [i \Delta_F(x - x') - i \Delta_F(0)]} \right. \\
&\quad \left. - \frac{g_s^4 C_A C_F}{24} \{ (i \Delta_F(x - x') - i \Delta_F(0)) (3i \Delta_F(x - x') - i \Delta_F(0)) \} + \mathcal{O}(g_s^6) \right]. \tag{10}
\end{aligned}$$

As expected, we have an additional term proportional to C_A at $\mathcal{O}(g_s^4)$. This term corresponds to the next to leading log term in the two-loop perturbative expansion of the massless quark propagator. Note that it has the desired factor $[i\Delta_F(x-x') - i\Delta_F(0)]$ which ensures chiral quark condensate is gauge invariant in QCD. We will not give the details of $\mathcal{O}(g_s^6)$ calculation. However, in the appendix, one can find the relevant identities, formulas and contractions to arrive at the following final result.

$$\begin{aligned} iS_{ij}^F(x, x') &= iS_{ij}^{0F}(x, x') \left[e^{g_s^2 C_F [i\Delta_F(x-x') - i\Delta_F(0)]} \right. \\ &\quad - \frac{g_s^4 C_A C_F}{(2!)(3!2!1!)} \{ [i\Delta_F(x-x') - i\Delta_F(0)] [3i\Delta_F(x-x') - i\Delta_F(0)] \} [1 + g_s^2 C_F (i\Delta_F(x-x') - i\Delta_F(0))] \\ &\quad \left. + \frac{g_s^6 C_F C_A^2}{(1!)(4!3!2!1!)} [i\Delta_F(x-x') - i\Delta_F(0)] [8(i\Delta_F(x-x'))^2 - 7(i\Delta_F(x-x'))(i\Delta_F(0)) + (i\Delta_F(0))^2] + \mathcal{O}(g_s^8) \right]. \end{aligned} \quad (11)$$

This expression suggests a possible start of a new next to leading log series at $\mathcal{O}(g_s^4)$ which might be summed up as follows:

$$\begin{aligned} iS_{ij}^F(x, x') &= iS_{ij}^{0F}(x, x') \left[e^{g_s^2 C_F [i\Delta_F(x-x') - i\Delta_F(0)]} \right. \\ &\quad - \frac{g_s^4 C_A C_F}{(2!)(3!2!1!)} \{ [i\Delta_F(x-x') - i\Delta_F(0)] [3i\Delta_F(x-x') - i\Delta_F(0)] \} e^{g_s^2 C_F [i\Delta_F(x-x') - i\Delta_F(0)]} \\ &\quad \left. + \frac{g_s^6 C_F C_A^2}{(1!)(4!3!2!1!)} [i\Delta_F(x-x') - i\Delta_F(0)] [8(i\Delta_F(x-x'))^2 - 7(i\Delta_F(x-x'))(i\Delta_F(0)) + (i\Delta_F(0))^2] + \mathcal{O}(g_s^8) \right]. \end{aligned} \quad (12)$$

We have verified the existence of this new exponential series till $\mathcal{O}(g_s^8)$. Eq. (4) is the $SU(N)$ modification of the LKF transformation for the fermion propagator in QED. Eq. (12) is the $\mathcal{O}(g_s^6)$ expansion of this transformation (though the exponentials are infinite order). We summarize below some key remarks regarding this transformation:

- For QED, $C_F = 1$, $g_s = e$, $C_A = 0$. On substituting these values, we recuperate the LKFT for the electron propagator [1].
- We find closed expression for the perturbative series in the color factor in the fundamental representation, i.e., C_F^n . However, a closed expression for the perturbative series in the color factor in the adjoint representation, namely C_A^n , is a harder nut to crack as it has several sources. This work is in progress and will be reported elsewhere.
- Recall the definition of the chiral quark condensate and the final expression for the quark propagator in an arbitrary covariant gauge, Eq. (12):

$$\begin{aligned} \langle \bar{\psi}\psi \rangle_\xi &= -\text{Tr} [S^F(x, x')]_{x'=x} \\ &= -\text{Tr} [S^{0F}(x, x')]_{x'=x} = \langle \bar{\psi}\psi \rangle_0 \end{aligned} \quad (13)$$

Hence the chiral quark condensate is manifestly a gauge invariant quantity in any $SU(N)$ theory. It owes itself to the fact that at least one color factor C_F appears at every order in perturbation theory in each term, and hence contributes an overall factor of $(i\Delta_F(x-x') - i\Delta_F(0))$ which guarantees the gauge invariance of the chiral quark condensate.

- To the order we have carried out our calculation, we observe the factors:

$$\begin{aligned} C_A &\Rightarrow \frac{1}{3!2!1!} [3i\Delta_F(x-x') - i\Delta_F(0)] \\ C_A^2 &\Rightarrow \frac{1}{4!3!2!1!} \left[8\{i\Delta_F(x-x')\}^2 + \{i\Delta_F(0)\}^2 \right. \\ &\quad \left. - 7\{i\Delta_F(x-x')\} \{i\Delta_F(0)\} \right]. \end{aligned} \quad (14)$$

We have verified that the first of this recurs at $\mathcal{O}(g_s^6)$ and $\mathcal{O}(g_s^8)$, and we conjecture it to become a coefficient of $e^{g_s^2 C_F [i\Delta_F(x-x') - i\Delta_F(0)]}$.

The local gauge transformation for the quark propagator has verifiable consequence to all orders in perturbation theory. For example, it implies multiplicative renormalizability of the massless quark propagator which we set out to discuss in the next section.

III. GAUGE TRANSFORMATION AND MULTIPLICATIVE RENORMALIZABILITY

The connection of local gauge transformation for a charged fermion with its leading $(\alpha\xi)^n$ or sub-leading $\alpha^n \xi^{n-1}$ perturbative expansion has been studied in detail for QED, [7]. For 4 space-time dimensions in massless QED, this expansion becomes a multiplicatively renormalizable power law for the wave function renormalization, [8]. Something similar takes place in QCD but the gluon self-interactions introduce additional series as we now see.

Two completely general Lorentz decompositions of quark propagator in momentum and coordinate space are, respectively:

$$\begin{aligned} S^F(p; \xi) &= \frac{F(p; \xi)}{\not{p} - \mathcal{M}(p; \xi)}, \\ S^F(x; \xi) &= \not{x} X(x; \xi) + Y(x; \xi), \end{aligned} \quad (15)$$

where we have set $x' = 0$ without the loss of generality and suppressed color indices. Let us start from the tree level quark propagator, i.e.,:

$$F(p; 0) = 1, \quad \mathcal{M}(p; 0) = 0. \quad (16)$$

We now proceed as follows. We Fourier transform this free quark propagator to the coordinate space and apply the gauge transformation law of Eq. (12) to find the quark Green function $S^F(x; \xi)$ in an arbitrary covariant gauge. Its inverse Fourier transform yields the quark propagator back in momentum space. To check the consequences of the transformation law, we restrict ourselves to the one loop perturbation theory where gluon self interactions do not contribute to the quark propagator and we can set $C_A = 0$. The Fourier transform of Eq. (16) is:

$$X(x; 0) = -\frac{1}{2\pi^2 x^4}, \quad Y(x; 0) = 0. \quad (17)$$

From Eq. (6), we deduce:

$$i\Delta_F(x) = \xi \mu^{4-d} \int_0^\infty \frac{d^d p}{(2\pi)^d} \frac{e^{-ip \cdot x}}{p^4}. \quad (18)$$

Here we have invoked d -dimensional integration to deal with ultraviolet divergences. Setting $d = 4 - 2\epsilon$,

$$i\Delta_F(x) = -\frac{\xi}{(4\pi)^2} \left[\frac{1}{\epsilon} + \gamma + 2\ln(\mu x) + \mathcal{O}(\epsilon) \right]. \quad (19)$$

As $\Delta_F(0)$ is divergent, we introduce a cut-off scale x_{\min} to write

$$i\Delta_F(x_{\min}) - i\Delta_F(x) = \lambda \ln \left(\frac{x^2}{x_{\min}^2} \right), \quad (20)$$

where $\lambda = \xi/(4\pi)^2$. Thus the LKF transformation yields:

$$S^F(x; \xi) = -\frac{\not{x}}{2\pi^2 x^4} \left(\frac{x^2}{x_{\min}^2} \right)^\nu, \quad (21)$$

where $\nu = C_F \alpha_s \xi/(4\pi)$ and $\alpha_s = g_s^2/(4\pi)$. Its inverse Fourier transform yields the quark propagator in momentum space

$$S^F(p; \xi) = \int d^4 x e^{ip \cdot x} S^F(x; \xi). \quad (22)$$

Thus

$$F(p; \xi) = \frac{1}{2^{2\nu}} \frac{\Gamma(1-\nu)}{\Gamma(2+\nu)} (p^2 x_{\min}^2)^\nu. \quad (23)$$

Note that in a cut-off regularization $x_{\min} \propto 1/\Lambda$. Recall from the previous section that the requirement of multiplicative renormalizability implies

$$\frac{F_R(p^2/\mu^2; \xi)}{F_R(k^2/\mu^2; \xi)} = \frac{F(p^2/\Lambda^2; \xi)}{F(k^2/\Lambda^2; \xi)}, \quad (24)$$

where the notation $F(p^2/\Lambda^2; \xi) \equiv F(p; \xi)$ has been used for the sake of clarity. We can choose $k^2 = \mu^2$ and impose the renormalization condition of restoring the tree level quark propagator at sufficiently large renormalization scale μ^2 . That is to say:

$$F_R(k^2/\mu^2; \xi)|_{k^2=\mu^2} = 1. \quad (25)$$

Thus

$$F_R(p^2/\mu^2; \xi) = \mathcal{Z}_2^{-1}(\mu^2; \Lambda^2) F(p^2/\Lambda^2; \xi), \quad (26)$$

where

$$\mathcal{Z}_2(\mu^2; \Lambda^2) = F(\mu^2/\Lambda^2; \xi). \quad (27)$$

Hence the LKFT ensure multiplicative renormalizability of the quark propagator. Taking into account the gluon self interactions would imply a more involved exponent ν beyond the leading order. The general structure of the LKFT and multiplicative renormalizability suggests the following form of the exponent:

$$\nu = f_0 C_F \alpha_s + f_1^a C_F^2 \alpha_s^2 + f_1^b C_F C_A \alpha_s^2 + \dots, \quad (28)$$

where $f_0, f_1^a, f_1^b \dots$ are determined through perturbative series of the quark propagator or its LKFT. For example, $f_0 = \xi/(4\pi)$. These contributions, in addition to being connected to the one loop quark and gluon propagators, are also intricately related to the quark-gluon vertex beyond the tree level which dictates these corrections through the quark gap equation. This is what we opt to discuss in the next section because any non perturbative construction of the quark-gluon vertex must ensure multiplicative renormalizability of the quark propagator. It becomes all the more relevant for the quenched version of the theory.

IV. QUARK-GLUON VERTEX AND LKF TRANSFORMATIONS

From the works in QED, we already know that the intricate structure of the quark gluon vertex dictates multiplicative renormalizability of the charged fermion and hence ensures LKF transformations for the 2-point function are satisfied. Brown and Dorey, [9], argue that an arbitrary construction of the electron-photon vertex does not satisfy the requirement of the multiplicative renormalizability. It was realized that neither the bare vertex nor the Ball-Chiu-vertex, [10], which satisfies the WFGTI, were good enough to fulfill the demands of multiplicative renormalizability. Since then, starting from

the pioneering work by Curtis and Pennington, [11], there have been improved attempts to incorporate the implications of LKFT in constructing a reliable electron-photon vertex ansatz, [12]. The need for the same in QCD was realized in the work by Bloch, who constructs a model truncation which preserves multiplicative renormalizability, and reproduces the correct leading order perturbative behavior through assuming non-trivial cancellations involving the full quark-gluon vertex in the quark self-energy loop, [13].

Note that the quark propagator beyond $\mathcal{O}(g_s^2)$ involves gluon self interactions. These interactions introduce the color factor C_A in the adjoint representation. For example, one-loop calculation of the transverse quark-gluon vertex, defined by $(k-p)_\mu \Gamma^\mu(k,p)=0$, where k and p are the incoming and outgoing fermion momenta, respectively, for the large separation of quark momenta scales, i.e., $k^2 \gg p^2$, reveals, [14]:

$$\begin{aligned}\Gamma_a^\mu(k,p) &= \left(C_F - \frac{1}{2}C_A\right) \frac{\alpha_s \xi}{8\pi k^2} \log\left(\frac{p^2}{k^2}\right) T^\mu(k,p), \\ \Gamma_n^\mu(k,p) &= C_A \frac{\alpha_s(3\xi-1)}{64\pi k^2} \log\left(\frac{p^2}{k^2}\right) T^\mu(k,p),\end{aligned}\quad (29)$$

where the subscripts a and n correspond to the abelian and non-abelian diagram, respectively. Vector $T^\mu(k,p)$ is defined as $T^\mu(k,p) = k^2 \gamma^\mu - k^\mu \not{k}$. Note that one expects terms of the type $\alpha_s \xi^2$ from the non-abelian diagram but these terms identically cancel out in this limit. The gauge parameter dependent piece of the quark gluon vertex would result in the following type of $\mathcal{O}(g_s^4)$ terms in the wave-function renormalization :

$$\begin{aligned}F(p^2/\mu^2; \xi) &= 1 + \alpha_s f_0 C_F \ln\left(\frac{p^2}{\mu^2}\right) \\ &+ \alpha_s^2 \left[\frac{f_0^2 C_F^2}{2} \ln^2\left(\frac{p^2}{\mu^2}\right) + C_F (f_1^a C_F + f_1^b C_A) \ln\left(\frac{p^2}{\mu^2}\right) \right].\end{aligned}$$

Note that to this order, the C_A terms would also be contributed by the gluon and ghost loop correction to the gluon propagator in the quark gap equation.

V. CONCLUSION

We derive the generalized LKFT for the quark propagator in an $SU(N)$ gauge field theory. We then expand this expression under local variation of gauge (4) to $\mathcal{O}(g_s^6)$ and, for some terms, to $\mathcal{O}(g_s^8)$. We find a closed expression for the perturbative series $C_F^n C_A^0$ and $C_F^{n-1} C_A$. We expect the same for terms higher in the order of C_A . However, as the color factor C_A is present in all different interactions including gluon 3- and 4-point self interactions, interactions involving ghosts as well as the quark-gluon interactions beyond one loop, it is not straightforward to disentangle different series in closed forms. However, work is in progress in this direction and we plan to report it elsewhere. Due to the explicit presence of the

overall factor of $(i\Delta_F(x-x') - i\Delta_F(0))$, the gauge invariance of the chiral quark condensate is guaranteed just as in QED, [15].

In the abelian approximation, we carry out the Fourier transform and obtain a multiplicatively renormalizable power law for the massless quark propagator. The general structure of the LKFT and the quark-gluon interactions at one loop order allow us to foresee the pattern of C_A terms of the non-abelian origin, which appear at the next to leading order level in the perturbative expression for the massless quark propagator.

The generalized LKFT for QCD also provide us with information on how gluon and ghost propagators as well as the three and four point vertices transform under the local variation of gauge. This work is in progress and will be reported elsewhere.

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Appendix:

The details and identities of color algebra can be found in different textbooks. However, for the sake of completeness and for a quick reference, we list most of the identities we used. The starting commutation and anti-commutation relations are:

$$[T_b, T_c] = if_{bcp} T_p, \quad \{T_b, T_c\} = \frac{1}{N} \delta_{bc} + d_{bcp} T_p.$$

Similarly, we use the identities:

$$\begin{aligned}f_{aab} &= 0, \quad d_{aab} = 0, \quad f_{bcp} f_{bcq} = N \delta_{pq}, \quad \delta_{pp} = N^2 - 1 \\ d_{bcp} d_{bcq} &= \left(N - \frac{4}{N}\right) \delta_{pq}, \quad \frac{1}{N} = -2 \left(C_F - \frac{1}{2}C_A\right).\end{aligned}$$

Now some simple products of two or three $SU(N)$ generators can be written as:

$$\begin{aligned}T_a T_a &= C_F, \quad T_a T_b T_a = \left(C_F - \frac{1}{2}C_A\right) T_b, \\ if_{abc} T_c T_a T_b &= -\frac{1}{2} C_A C_F.\end{aligned}$$

Moreover,

$$\begin{aligned}T_a T_b T_a T_b &= C_F (C_F - \frac{1}{2}C_A), \\ T_a T_b T_c T_a T_b T_c &= C_F (C_F - C_A) (C_F - \frac{1}{2}C_A), \\ T_a T_b T_c T_b T_a T_c &= C_F (C_F - \frac{1}{2}C_A)^2, \\ T_a T_b T_a T_c T_b T_c &= C_F (C_F - \frac{1}{2}C_A)^2, \\ T_a T_b T_c T_a T_c T_b &= C_F (C_F - \frac{1}{2}C_A)^2, \\ T_a T_b T_c T_b T_c T_a &= C_F^2 (C_F - \frac{1}{2}C_A).\end{aligned}$$

$$\begin{aligned}
(i\Delta_F(0))\delta_{ab}[\langle 0|\mathcal{T}\{\varphi_c(x)\varphi_d(x)\varphi_e(x)\varphi_f(x)\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^3[C_F^2(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(0))\delta_{ac}[\langle 0|\mathcal{T}\{\varphi_b(x)\varphi_d(x)\varphi_e(x)\varphi_f(x)\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^3[C_F(C_F - \frac{1}{2}C_A)(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(0))\delta_{ad}[\langle 0|\mathcal{T}\{\varphi_b(x)\varphi_c(x)\varphi_e(x)\varphi_f(x)\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^3[C_F^3 + C_F(C_F - \frac{1}{2}C_A)(2C_F - \frac{3}{2}C_A)], \\
(i\Delta_F(0))\delta_{ae}[\langle 0|\mathcal{T}\{\varphi_b(x)\varphi_c(x)\varphi_d(x)\varphi_f(x)\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^3[C_F(C_F - \frac{1}{2}C_A)(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(0))\delta_{af}[\langle 0|\mathcal{T}\{\varphi_b(x)\varphi_c(x)\varphi_d(x)\varphi_e(x)\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^3[C_F^2(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(x-x'))\delta_{ab}[\langle 0|\mathcal{T}\{\varphi_c(x')\varphi_d(x')\varphi_e(x')\varphi_f(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^2(i\Delta_F(x-x'))[C_F^2(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(x-x'))\delta_{ac}[\langle 0|\mathcal{T}\{\varphi_b(x')\varphi_d(x')\varphi_e(x')\varphi_f(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^2(i\Delta_F(x-x')) \\
&\times [C_F(C_F - \frac{1}{2}C_A)(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(x-x'))\delta_{ad}[\langle 0|\mathcal{T}\{\varphi_b(x')\varphi_c(x')\varphi_e(x')\varphi_f(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^2(i\Delta_F(x-x')) \\
&\times [C_F^3 + C_F(C_F - \frac{1}{2}C_A)(2C_F - \frac{3}{2}C_A)], \\
(i\Delta_F(x-x'))\delta_{ae}[\langle 0|\mathcal{T}\{\varphi_b(x')\varphi_c(x')\varphi_d(x')\varphi_f(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^2(i\Delta_F(x-x')) \\
&\times [C_F(C_F - \frac{1}{2}C_A)(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(x-x'))\delta_{af}[\langle 0|\mathcal{T}\{\varphi_b(x')\varphi_c(x')\varphi_d(x')\varphi_e(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^2(i\Delta_F(x-x'))[C_F^2(3C_F - \frac{1}{2}C_A)] \\
(i\Delta_F(0))\delta_{ab}[\langle 0|\mathcal{T}\{\varphi_c(x')\varphi_d(x')\varphi_e(x')\varphi_f(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))^3[C_F^2(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(x-x'))\delta_{ac}[\langle 0|\mathcal{T}\{\varphi_b(x)\varphi_d(x')\varphi_e(x')\varphi_f(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))(i\Delta_F(x-x'))^2 \\
&\times [C_F(C_F - \frac{1}{2}C_A)(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(x-x'))\delta_{ad}[\langle 0|\mathcal{T}\{\varphi_b(x)\varphi_c(x')\varphi_e(x')\varphi_f(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))(i\Delta_F(x-x'))^2 \\
&\times [C_F^3 + C_F(C_F - \frac{1}{2}C_A)(2C_F - \frac{3}{2}C_A)], \\
(i\Delta_F(x-x'))\delta_{ae}[\langle 0|\mathcal{T}\{\varphi_b(x)\varphi_c(x')\varphi_d(x')\varphi_f(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))(i\Delta_F(x-x'))^2 \\
&\times [C_F(C_F - \frac{1}{2}C_A)(3C_F - \frac{1}{2}C_A)], \\
(i\Delta_F(x-x'))\delta_{af}[\langle 0|\mathcal{T}\{\varphi_b(x)\varphi_c(x')\varphi_d(x')\varphi_e(x')\}|0\rangle]T_aT_bT_cT_dT_eT_f &= (i\Delta_F(0))(i\Delta_F(x-x'))^2[C_F^2(3C_F - \frac{1}{2}C_A)].
\end{aligned}$$

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